

EQUILIBRIUM IN REAL OPTIONS

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Abstract

We determine the equilibrium timing of a capacity expansion in a market with many potential investors. The problem is solved with two different methods. First we solve the problem from the point of view of a central planner, using a theorem previously derived in Kobila (1991). This method also gives the monopoly solution. Secondly we derive the market equilibrium by using a Skorohod stochastic differential equation. The equilibrium turns also out to be a subgame perfect Nash equilibrium with two or more players. Thus even the oligopoly solution is socially efficient. Finally we discuss risk adjustment.

Keywords: *Real Options, Investment, Local time, Stochastic Control*

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1 Introduction

The literature on optimal irreversible investment under uncertainty has been rapidly growing over the last decade. Recently developed stochastic optimization techniques offer new insight into the combined effect of uncertainty and irreversibility. These methods were first applied to financial option value problems, but have later been applied to real options, i. e. investments in physical capital. Important contributions include Brennan and Schwartz (1985) and McDonald and Siegel (1986). These applications consider the investment problem of an agent having only one investment opportunity.

Pindyck (1988) solves the problem of optimal capacity choice and capacity expansion under uncertainty of future demand and irreversibility of investment, given specific functional forms. He explicitly states the problem of evaluating a marginal unit of capacity as an option value problem and establishes the link to financial option value techniques. A more general solution to the same problem is obtained in Kobila (1991), using a direct approach based on a generalization of the Hamilton-Jacobi-Bellman equation.

In this paper we generalize the valuation of real options to the case where the investment decision influences the market price. The outline of the paper is as follows. In Section 2 we apply the results from Kobila (1991) to derive the socially optimal investment decision — maximizing total willingness to pay when the demand function is subject to random shifts. We show that the optimal investment strategy is to invest once the price of the produced commodity reaches a specific level, the "trigger price". In section 3 we show that the optimal investment strategy in the monopoly case is of the same form as the socially optimal strategy, but with a higher trigger price.

In Section 4 we assume that the investments are undertaken by profit-maximizing firms, and derive the equilibrium solution. We will assume that all investors can expand their capacity infinitely fast at a constant unit cost per capacity unit. With this assumption it turns out that in equilibrium the real investment option has no value. This is not surprising, when all investors have an infinite amount of the same option, the market value of the option must be zero.

The equilibrium solution is of course socially efficient, hence the trigger price is the same as the one found in section 2. In section 5 we will also show that the solution is a subgame perfect Nash-equilibrium in a game with a small number of players. This requires some extra condition to make sure that full capacity utilization always will be a Bertrand equilibrium. With these extra conditions the oligopoly solution is socially efficient as well.

Risk adjustment is discussed in section 6. According to the Consumption based Capital Asset Pricing Model (CCAPM) a risk free discount rate is justified if the price of the product is uncorrelated with the market portfolio. If the price is correlated with the market portfolio, it turns out that the appropriate risk adjustment consists in giving the expected growth rate of the output price a

constant adjustment, and use the risk-free discount rate.

In section 7 we will demonstrate a connection to traditional real option theory. The traditional option value of an investment opportunity is equal to the difference between the value of a unit capacity with non-reflected and reflected price processes.

2 Socially optimal investment with risk neutrality

In order to find the socially optimal investment strategy, the planner seeks to maximize the expected discounted welfare function less investment cost. The objective function is given by

$$H(t, q, k) = \sup_{u_t} E^{t,q,k} \left\{ \int_t^\infty (W(Q_s, K_s) - Cu_s) e^{-rs} ds \right\}. \quad (1)$$

Here $W(Q_t, K_t)$ is the total willingness to pay function depending on capacity K_t , and the stochastic process Q_t . That capacity and not actual supply enters W is justified by assuming full capacity utilization. Since we specify no variable costs, efficiency requires full capacity utilization. The scalars corresponding to the processes Q_t and K_t are denoted by q and k . The optimal value function H depends on time as well as on the two state variables. u_t denotes investments and C is the investment cost which is assumed constant. The discount rate is r and is here assumed constant.

The stochastic process Q_t is assumed to be exogenous and is specified as a geometric Brownian motion,

$$dQ_t = \eta Q_t dt + \sigma Q_t dB_t, \quad (2)$$

where η is the expected growth rate and σ^2 is the expected variance of the increments over a unit time interval, conditional on Q_t . Q_t is a parameter in the demand function. It may be interpreted as total income for the consumers of the good or as a parameter of changing tastes. Note that since Q_t is exogenous, the demand is independent of the production in the sector we study. This implies that we implicitly assume that the sector under study is too small to contribute significantly to the total income in the economy.

We assume that the capacity can be increased instantaneously, and that the investments are irreversible. Capacity expansion is given by

$$dK_t = u_t dt, \quad (3)$$

We will allow the rate of investment u_t to be infinite, or rather allow the process K_t singular components. In this case (1) is no longer the correct formulation, and we will replace it with

$$H(t, q, k) = \sup_{u_s} E^{t,q,k} \left\{ \int_t^\infty W(Q_s, K_s) e^{-rs} ds - C e^{-rs} dK_s \right\}. \quad (4)$$

The solution to the stochastic control problem (4) when W is linear (affine) in q is given by Theorem 3 in Kobiła (1991). In the notation of this paper, we have the following theorem.

Theorem 1 *Let Q_t be a geometric Brownian motion as given by (2). Suppose that the function $W(q, k)$ is linear (affine) in q , i.e.,*

$$W(q, k) = qF(k) - v(k), \quad (5)$$

with $F'(k) > 0$ and $F'' < 0$. Suppose that we only consider controls u_t such that $K_t \leq k_{\max}$. Let γ_1 be the positive root and γ_2 the negative root of the characteristic equation

$$\frac{1}{2}\sigma^2\gamma^2 + (\eta - \frac{1}{2}\sigma^2)\gamma - r = 0. \quad (6)$$

Then the solution to

$$H(t, q, k) = \sup_{u_s} E^{t,q,k} \left[\int_t^\infty W(Q_s, K_s) e^{-rs} ds - C e^{-rs} dK_s \right], \quad (7)$$

is given by $H = h$, where

$$h(t, q, k) = \frac{2e^{-rt}}{(\gamma_1 - \gamma_2)\sigma^2} \left[-q^{\gamma_1} \int_\infty^q \frac{\bar{W}(s, k)}{s^{\gamma_1+1}} ds + q^{\gamma_2} \int_0^q \frac{\bar{W}(s, k)}{s^{\gamma_2+1}} ds \right]. \quad (8)$$

$\bar{W}(q, k)$ is given by

$$\bar{W}(q, k) = \begin{cases} W(q, k) & \text{for } k > \phi(q) \\ W(q, \phi(q)) - rC \cdot (\phi(q) - k) & \text{otherwise} \end{cases}, \quad (9)$$

where $k = \phi(q)$ denotes the inverse function of $q = \psi(k)$, and ψ is determined by the equation

$$C = \frac{2}{\sigma^2\gamma_1} \psi(k)^{\gamma_2} \int_0^{\psi(k)} \frac{W'_k(s, k)}{s^{\gamma_2+1}} ds. \quad (10)$$

The optimal capacity process is

$$K_t^* = K_0 \vee \max_{s \leq t} (\phi(Q_s)) \quad (11)$$

Remark The corresponding optimal control is, heuristically

$$u^*(t, q, k) = \begin{cases} 0 & \text{for } q < \psi(k) \text{ or } k > k_{\max} \\ \infty & \text{otherwise} \end{cases}. \quad (12)$$

The optimal investment strategy given by (12) – resulting in a stochastic process K_t^* of optimal capacity – represents a singular solution where optimal investment is either 0 or ∞ . Under reasonable assumptions we cannot have $u_t = \infty$ on a time interval of positive length, hence, the solution will be of the following form: There exists an open subset \mathcal{A} of the (q, k) -plane such that $u_t = \infty$ if and only if $(Q_t, K_t) \in \bar{\mathcal{A}}$. If the process (Q_t, K_t^*) starts within \mathcal{A} it will immediately be thrown out of \mathcal{A} , and it is impossible for the process to enter \mathcal{A} from the outside. The K_t component of the process will jump vertically to the boundary $\partial\mathcal{A}$ of \mathcal{A} if (Q_t, K_t^*) starts inside \mathcal{A} . In Kobila (1989) we have shown that (Q_t, K_t^*) is a Markov process with horizontal movements outside \mathcal{A} and vertical reflection on $\partial\mathcal{A}$.

Proof (sketch) A complete proof of this theorem is given in Kobila (1991). The main idea is first to prove an extension of the Hamilton–Jacobi–Bellman (HJB) equation (see Øksendal (1989)) where equality is required only for $q \leq \psi(k)$. Guessing that H is of the form $H(t, q, k) = G(q, k)e^{-rt}$, the HJB equation can be written:

$$W(q, k) - rG + \eta q G'_q + \frac{1}{2} \sigma^2 q^2 G''_{qq} \leq 0, \quad (13)$$

with equality for $q \leq \psi(k)$. It is easy to verify that $g(q, k) = h(t, q, k)e^{rt}$ satisfy this equation. A separate argument must be used to prove that the expected net present profit using the control u^* in fact gives the supremum of expected net present profit over all $u < \infty$. QED

In the theorem the cost C per unit investment is allowed to depend upon the amount of capacity already installed. In our application we will consider the case where C is constant. For the details of the proof and the intuitive explanation of the singular investment strategy (12), see Kobila (1990).

Integrating (10) and using the relation between the roots of the characteristic equation

$$\gamma_1 \gamma_2 = -\frac{2r}{\sigma^2}, \quad (14)$$

we find, with $W(q, k)$ as given in (5), and C independent of k :

$$rC = -\gamma_2 \psi(k)^{\gamma_2} \int_0^{\psi(k)} \frac{W'_k(s, k)}{s^{\gamma_2+1}} ds = \frac{\gamma_2}{\gamma_2 - 1} \psi(k) \lambda'(k) - v'(k), \quad (15)$$

(15) implicitly characterizes the optimal investment strategy by the relation $q = \psi(k)$.

We assume that the willingness to pay function is subject to random fluctuations represented by the stochastic process Q_t . In order to obtain an explicit solution, we assume that W is a linear function of q ,

$$W(q, k) = F(k)q, \quad (16)$$

with $F'(k) > 0$ and $F''(k) < 0$. The equilibrium price for the optimally determined capacity is the marginal willingness to pay:

$$P_t = F'(K_t)Q_t. \quad (17)$$

Using this equation, we can find the critical price level p^* at which investment takes place. From (12) we have that the critical level for investment is $q = \psi(k)$, or in terms of (17),

$$p^* = \psi(k)F'(k). \quad (18)$$

From (14) and another relation between the roots of the characteristic equations

$$\gamma_1 + \gamma_2 = 1 - \frac{2\eta}{\sigma^2}, \quad (19)$$

we derive the additional relation

$$\frac{\gamma_2 - 1}{\gamma_2} = \frac{\gamma_1}{\gamma_1 - 1} \cdot \frac{r - \eta}{r}.$$

Using (18) in (15) (with $\lambda = F$ and $v = 0$) we get

$$p^* = \frac{\gamma_2 - 1}{\gamma_2} rC = \frac{\gamma_1}{\gamma_1 - 1} \frac{r - \eta}{r} rC. \quad (20)$$

Hence the socially optimal strategy is to invest as the market price exceeds the critical level p^* . Note that since $\gamma_2 < 0$ we have $p^* > rC$, and hence it is not optimal to invest once the price reaches the long term marginal cost.

3 The monopoly solution

A monopolist maximizing expected net present value of future profit has to solve the problem

$$H(t, q, k) = \sup_{u_s} E^{t,q,k} \left\{ \int_t^\infty \pi(Q_s, K_s) e^{-rs} ds - C e^{-rs} dK_s \right\}, \quad (21)$$

where $\pi(Q_t, K_t)$ denote the monopoly profit. Since there are no variable cost in this model, the profit is given as

$$\pi(q, k) = \max_{x \leq k} x F'(x) q, \quad (22)$$

where x is the quantity produced by the monopolist at time t , and where the output price is derived from the market clearing condition (17). Note that the optimal X_t is independent of Q_t , hence if optimal X_t is less than K_t at some time

t it will never be optimal to invest any further at any future time $s > t$. When demand is elastic, i.e.,

$$\frac{F''(x)x}{F'(x)} > -1, \quad (23)$$

for all x , we get that $X_t = K_t$ is optimal and hence optimal profit is $K_t F'(K_t) Q_t = f(K_t) Q_t$.

Note that the only difference between the monopoly problem and the problem of finding the social optimum, is that F is replaced by f . Thus, assuming f is concave we solve the problem faced by the monopolist by replacing F by f in the solution (18) and (20)¹. Hence, the optimal solution is to invest once marginal profit, $f'(K_t) Q_t$, exceeds p^* . Since

$$f'(K_t) Q_t = F'(K_t) Q_t + K_t F''(K_t) Q_t = P_t + K_t F''(K_t) Q_t, \quad (24)$$

the investment criterion can be reformulated as to invest when:

$$P_t \geq p^* - K_t F''(K_t) Q_t > p^*. \quad (25)$$

The optimal investment strategy is of the same form as the socially optimal investment strategy, but with a higher trigger price. Thus, the monopolist will underinvest and hence produce less than the socially optimal level of production.

4 The market equilibrium

In this section we will derive the market equilibrium. Like in an Arrow-Debreu equilibrium we will assume that the agents know the equilibrium price that will prevail at any time and any state of the world. It will turn out that with risk neutrality it would have been sufficient to assume rational expectations.

The problem of determining the optimal time to exercise a real option (i. e. undertake an investment in physical capital) is different from the problem of when to exercise a financial option. The reason is that exercising a real option affects the equilibrium price. When an agent exercises a financial call option, the asset is bought and sold at the same moment in time, and the asset price is not influenced. On the other hand, when a real option is exercised, additional production capacity is acquired, the supply of the product is increased and the market price is influenced.

Let us return to the market considered in section 2. Since the equilibrium is socially efficient we know from section 2 that in equilibrium the agent waits for a trigger price \hat{p} , and starts investing once the price exceeds this level, i.e. once $P_t \geq \hat{p}$. By social efficiency we know that $\hat{p} = p^*$, but we will derive the level of \hat{p} by an independent argument.

¹The use of theorem 1 requires that f is increasing and concave. The fact that f is increasing is a consequence of the assumption about the elasticity, but some extra requirement on F are implicit in the assumption that f is concave.

If P_t is infinitesimally higher than \hat{p} , all agents want to exercise their options. The increase in production capacity would immediately press the price down to \hat{p} . Hence, in the case of many identical producers there exists an upper bound on the price process and the price cannot exceed \hat{p} .

To describe the price process mathematically in this case we argue as follows:

As long as the price is less than \hat{p} the process P_t should behave like the geometric Brownian motion Q_t , since K_t is constant there. This follows by using Itô's lemma on (17):

$$\begin{aligned} dP_t &= F'(K_t)dQ_t + F''(K_t)Q_t dK_t \\ &= F'(K_t)\eta Q_t dt + F'(K_t)\sigma Q_t dB_t \\ &= \eta P_t dt + \sigma P_t dB_t, \end{aligned} \quad (26)$$

if $P_t < \hat{p}$. However, as soon as P_t reaches the value \hat{p} the instantaneous increase in production capacity pushes the price back to the level below \hat{p} again. This means that P_t for any t should satisfy a stochastic differential equation of the form:

$$dP_t = \eta P_t dt + \sigma P_t dB_t - \chi_{\{\hat{p}\}}(P_t) d\xi_t; \quad P_0 = p < \hat{p}, \quad \xi_0 = 0, \quad (27)$$

for some non-decreasing stochastic process ξ_t which only increases for

$$t \in \Lambda = \Lambda(\omega) = \{t \geq 0; P_t = \hat{p}\}. \quad (28)$$

Moreover, we require that $P_t \leq \hat{p}$ always and that $\Lambda(\omega)$ has zero Lebesgue measure, a.s. (the last condition is reasonable since we cannot have $u_t^* = \infty$ on a set of t -values of positive measure (see Theorem 1).)

The equation (27) is called a Skorohod stochastic differential equation (in the two unknown processes P_t and ξ_t). According to a theorem due to Skorohod, Watanabe, Anderson and Grey (see Freidlin (1985), Th. 6.1) there exists under the given condition a unique solution (P_t, ξ_t) of this equation. The process P_t is called *the reflection* of the process Q_t at \hat{p} , and ξ_t is called *the local time* of P_t at \hat{p} .

To derive the market equilibrium we need to know the value of a unit capacity when the price is reflected at a given level \hat{p} . We will let the discount rate depend upon the price. I.e. we must know:

$$G(p) = E^p\left[\int_t^\infty P_s \exp\left(-\int_t^s r(P_x) dx\right) ds\right], \quad (29)$$

where P_s is the reflection of Q_t at \hat{p} .

From a theorem in Freidlin (1985, Th. 5.2), we have that:

Theorem 2 Suppose $g \in C^2$ solves the equation:

$$-r(p)g(p) + \eta p g'(p) + \frac{1}{2}\sigma^2 p^2 g''(p) = -p, \quad (30)$$

with $g(0) = 0$, and $g'(\hat{p}) = 0$ then

$$g(p) = E^p[\int_t^\infty P_s \exp(-\int_t^s r(P_x)dx)ds], \quad (31)$$

where P_s is the reflected process given in (27).

An intuitive explanation for the boundary condition $g'(\hat{p}) = 0$ is that as $P_t = \hat{p}$, it does not matter whether the nonreflected process increases or decreases, the effect on the reflected process is equivalent, and hence $g'(\hat{p}) = 0$.

In this section we will consider the case that r is independent of p . In the next section we argue that under the assumption of CAPM we can reduce the general problem to this specific case. The general solution to (30) with constant r is (using the first boundary condition)

$$g(p) = \frac{1}{r - \eta} p + \nu p^{\gamma_1}, \quad (32)$$

where γ_1 is the positive root of the characteristic equation, and ν is a constant. The second boundary condition gives the equation:

$$\frac{\hat{p}}{r - \eta} + \nu \gamma_1 \hat{p}^{\gamma_1} = 0. \quad (33)$$

For \hat{p} to constitute an equilibrium, it must not be profitable for anyone to deviate from the strategy

$$I_{it} = \begin{cases} 0 & \text{for } P_t < \hat{p} \\ \infty & \text{otherwise.} \end{cases} \quad (34)$$

where I_{it} is the rate of investment for agent i at time t . If the present value of the investment is zero for $P_t = \hat{p}$ this will be the case. I.e. it will not be profitable for anyone to deviate from the strategy (34): investing for $P_t < \hat{p}$ gives a present value less than zero, and the policy to wait for prices strictly higher than \hat{p} will not give any investment. Hence the last condition identifying the equilibrium level of \hat{p} is that the expected present value of future profit from unit capacity at $P_t = \hat{p}$, must be equal to the unit cost of investment. When r is independent of P , the expected present value is given by (32), and we get:

$$g(\hat{p}) = \frac{\hat{p}}{r - \eta} + \nu \hat{p}^{\gamma_1} = C. \quad (35)$$

Combining (33) and (35) we get:

$$\hat{p} = \frac{\gamma_1}{\gamma_1 - 1} \cdot \frac{r - \eta}{r} rC = p^*. \quad (36)$$

The last equality was derived from (20).

5 The oligopoly solution

Suppose that the market consists of a small number of players. Assume for a moment that it is optimal for each player to produce at full capacity at all points of time. Then the argument from the previous section will apply even with only two players. Nobody can gain from deviating from the investment strategy of infinite rate of investment for $P_t > p^*$.

It remains to justify the assumption that it is optimal for the players to produce at full capacity. If player i charges a price p_i where $p_i > p_j$ for all $j \neq i$, he faces a demand $D(p_i) - Q_{-i}$, where Q_{-i} is the total supply from all firms except i , and $D(\cdot)$ is the market demand. Now suppose that all players charge a price equal to the market clearing price, and hence produces at full capacity utilization. Let $s_i = \frac{K_i}{K}$ be i 's market share at full capacity utilization. Then a straightforward calculations shows that i cannot increase profit by charging a higher price if

$$\frac{F''(x)x}{F'(x)} > -\frac{1}{s_i}. \quad (37)$$

Since $\frac{1}{s_i} \geq 1$, this inequality is obviously satisfied if (23) is satisfied. In this case, it is even optimal for the monopolist to produce at full capacity, hence in this case it will obviously be optimal for the oligopolists as well.

Note, however, that the market shares will change over time. Thus, to study the equilibrium in the case that (23) is not satisfied, we would have to specify more carefully the *relative* speed of investment. There are special cases where the oligopoly solution is still equal to the market equilibrium even for elasticities less than -1. To take an example consider a market with the players with equal market share, and equal speed of investment, and where the elasticity is higher than -2. A throughout analysis of the cases with elasticities less than -1 is, however, beyond the scope of this paper.

6 Risk adjustment

In the previous section we have assumed risk neutrality. This is a very strong assumption, and we wish to relax it. Since the market equilibrium is an Arrow-Debreu equilibrium that is known to be socially efficient, there is thus no need to consider risk adjustment for the social optimum and the market equilibrium separately. It turns out to be easier to consider the market equilibrium.

Note that in the market equilibrium the firms are indifferent between investing and not investing as $P_t = \hat{p}$. It is thus optimal for a single firm not to invest at any time. Consider a firm with a unit capacity that follows this strategy. What is the market value of a share in this firm?

Note that equation (30) in theorem 2 allows r to depend upon p . If the equilibrium price can be found by discounting the future revenues with a discount rate $r(P_t)$ depending only upon P_t , the value of the firm will be $G_t = g(P_t)$ where $g(P_t)$ is the function given in theorem 2. Suppose for a moment that this is the case. Then G_t will be an Itô process, even though P_t has a singular component. The singularity at $P_t = \hat{p}$ has no effect on G_t since $g'(\hat{p}) = 0$. Since G_t is an Itô process the equilibrium rate of return is determined in the Consumption based Capital Asset Pricing Model (CCAPM).

In CCAPM equilibrium requires that the return r on an asset with price S_t with diffusion term $\sigma S_t dB_t$ is equal to:

$$r = r_0 + \rho_{Bm} \frac{\sigma}{\sigma_m} (r_m - r_0) = r_0 + \mu \sigma, \quad (38)$$

where ρ_{Bm} is the correlation between B_t and the market portfolio, σ_m and r_m is standard deviation and return on the market portfolio. Finally

$$\mu = \frac{\rho_{Bm}}{\sigma_m} (r_m - r_0).$$

To apply this result we need to know the diffusion term of the process G_t . By the generalized Itô rule the diffusion term of G_t is

$$\frac{g'(P_t)}{g(P_t)} \sigma G_t dB_t.$$

Hence the required rate of return in the CCAPM depends only upon P_t as assumed, and the required rate of return is:

$$r(p) = r_0 + \mu \sigma p \frac{g'(p)}{g(p)}. \quad (39)$$

Inserting (39) into (30) gives

$$-r_0 g(p) + \hat{\eta} p g'(p) + \frac{1}{2} \sigma^2 p^2 g''(p) = -p, \quad (40)$$

where

$$\hat{\eta} = \eta - \mu \cdot \sigma. \quad (41)$$

Thus the appropriate risk adjustment is equivalent to adjusting the growth rate of Q_t from η to $\hat{\eta}$.

By a similar argument we can show the same result in the monopoly case. We conclude that in both these cases the appropriate risk adjustment is to adjust the expected growth rate of the price process, and use the risk-free rate of return. Similar results are well known in option pricing theory. (See Harrison and Kreps (1979), Harrison and Pliska (1981) and Cox, Ingiersoll and Ross (1985).) For real options a similar result was derived in McDonal and Siegel (1986)

7 The relation between the reflected and the unreflected processes

Let \bar{P}_t denote the price process reflected at p^* , and let P_t denote the unreflected version of the same process. From previous studies of real options (for instance McDonald and Siegel (1986)) we know the optimal time of investment if the product price is not reflected. It turns out to be optimal to delay investment until P_t reaches the level p^* , which is the market equilibrium reflection price level derived in this paper. As a consequence of this observation we will show that the difference in value of a unit capacity with prices P_t and \bar{P}_t is equal to the value of a traditional option.

To be more precise, let $V(p)$ be the value of a traditional real option given that the price process starts in p , i.e.,

$$V(p) = E^p \left\{ \int_{\tau}^{\infty} P_t e^{-rt} dt - C e^{-r\tau} \right\} \quad (42)$$

where τ is the optimal timing of the investment. The value, $U(p)$ of a unit capacity with nonreflected prices is:

$$U(p) = E^p \left\{ \int_0^{\infty} P_t e^{-rt} dt \right\} = \frac{p}{r - \eta}. \quad (43)$$

Let $G(P)$ denote the value of a unit capacity when the prices is reflected. Our claim is then that:

$$U(p) - G(p) = V(p). \quad (44)$$

The claims is derived as follows,

$$\begin{aligned} U(p) - G(p) &= E^p \left\{ \int_0^{\infty} (P_t - \bar{P}_t) e^{-rt} dt \right\} \\ &= E^p \left\{ \int_{\tau}^{\infty} (P_t - \bar{P}_t) e^{-rt} dt \right\} \\ &= E^p \left\{ \int_{\tau}^{\infty} P_t e^{-rt} dt - C e^{-r\tau} \right\} - E^p \left\{ \int_{\tau}^{\infty} \bar{P}_t e^{-rt} dt - C e^{-r\tau} \right\} \quad (45) \\ &= E^p \left\{ \int_{\tau}^{\infty} P_t e^{-rt} dt - C e^{-r\tau} \right\} \\ &= V(p). \end{aligned}$$

The second equation follow from the fact that $P_t = \bar{P}_t$ until the process reaches the reflection level. In the third equation we just add and substract the same amount $C e^{-r\tau}$. The forth equation follows from the equilibrium condition that future value of production must be equal to investment cost for $\bar{P}_t = p^*$.

To interpret this result we compare a market where the price of output is exogenously given as in traditional real option theory, with the market where everybody has an infinite amount of the option. In this last case the option value has to be zero, since there is no scarcity in options. The output price is reflected exactly at the level wich makes the value of the real option zero. The previous result states that the reflection reduces the value of an already installed unit of capacity with the same amount.

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